Optimal Stopping with Rank-Dependent Loss

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Abstract

For τ a stopping rule adapted to a sequence of n iid observations, we define the loss to be $\mathbb{E}\,[q(R_\tau)]$, where R_j is the rank of the jth observation, and q is a nondecreasing function of the rank. This setting covers both the best choice problem with $q(r) = \mathbf{1}(r > 1)$, and Robbins' problem with $q(r) = r$. As $n \to \infty$ the stopping problem acquires a limiting form which is associated with the planar Poisson process. Inspecting the limit we establish bounds on the stopping value and reveal qualitative features of the optimal rule. In particular, we show that the complete history dependence persists in the limit, thus answering a question asked by Bruss [3] in the context of Robbins' problem.

Keywords: optimal stopping, Robbins' problem, best-choice problem, planar Poisson process

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1. Introduction Let $X_1, \dots, X_n$ be a sequence of iid observations, sampled from the uniform distribution on $[0, 1]$ (in the setup of this paper this assumption covers the general case of arbitrary continuous distribution). For $j \in [n] := \{1, \dots, n\}$ define *final ranks* as

$$R_j = \sum_{k=1}^{n} \mathbf{1}(X_k \le X_j),$$

so $(R_1, \dots, R_n)$ is an equiprobable permutation of $[n]$. Let $q : \mathbb{N} \to \mathbb{R}_+$ be a nondecreasing loss function with $q(1) < q(\infty) := \sup q(r)$. In 'secretary problems' [20] one is typically interested in the large-n behaviour of the minimum risk

$$V_n(\mathcal{T}_n) = \inf_{\tau \in \mathcal{T}_n} \mathbb{E}[q(R_\tau)], \tag{1}$$

where $\mathcal{T}_n$ is a given class of stopping rules with values in $[n]$. Two classical loss functions are

(i) $q(r) = \mathbf{1}(r > 1)$, for the best-choice problem of maximising the probability of stopping at the minimum observation $X_{n,1} := \min(X_1, \dots, X_n)$,

(ii) $q(r) = r$, for the problem of minimising the expected rank.

Many results are available for the case where $\mathcal{T}_n$ in (1) is the class $\mathcal{R}_n$ of *rank rules*, which are the stopping rules adapted to the sequence of *initial ranks*

$$I_j = \sum_{k=1}^{j} \mathbf{1}(X_k \le X_j) = \sum_{k=1}^{j} \mathbf{1}(R_k \le R_j) \qquad (j \in [n]),$$

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see [8, 9, 10]. By independence of the initial ranks, the optimal decision to stop at the jth observation depends only on I_j. The limiting risk $V_\infty(\mathcal{R}) := \lim_{n\to\infty} V_n(\mathcal{R}_n)$ has interpretation in terms of a continuous-time stopping problem [10]. Explicit formulas for $V_\infty(\mathcal{R})$ are known in some cases, for bounded and unbounded q, including the two classical loss functions and their generalisations [2, 7, 8, 16, 17].

Much less explored are the problems where $\mathcal{T}_n$ is the class $\mathcal{F}_n$ of all stopping rules adapted to the natural filtration $(\sigma(X_1, \dots, X_j),\ j \in [n])$. The principal difficulty here is that, for general q, the decision to stop on X_j must depend not only on X_j but also on the full vector $(X_{j-1,1}, \dots, X_{j-1,j-1})$ of order statistics of $X_1, \dots, X_{j-1}$. In this sense, the optimal rule is *fully history-dependent*. Specifically, the $\mathcal{F}_n$-optimal rule has the form

$$\tau_n = \min\{j : X_j < h_j(X_{j-1,1}, \dots, X_{j-1,j-1})\} \tag{2}$$

(with $h_{n,1} = \text{const},\ h_{n,n} = 1$), where $(h_{n,j},\ j \in [n])$ is a collection of functions with certain monotonicity properties. The dependence on history is reducible to the first $m-1$ order statistics if q is *truncated* at m: $q(r) = q(m)$ for $r \geq m$, but even then the analytical difficulties are severe. The asymptotic value $V_\infty(\mathcal{F}) := \lim_{n\to\infty} V_n(\mathcal{F}_n)$ is known explicitly only for the best-choice problem (hence for any q truncated at $m = 2$), see [12] for the formula and history. *Robbins' problem* is the problem (1) with $\mathcal{T}_n = \mathcal{F}_n$ and the linear loss function $q(r) = r$, see [1, 3, 4, 5].

The full history dependence makes explicit analysis of the $\mathcal{F}_n$-optimal rule hardly possible, thus it is natural to seek for tractable smaller classes of rules, with some kind of reduced dependence on the history. Of course, the rank rules is one of such classes, and the optimal rule in $\mathcal{R}_n$ is also of the form (2), with the special feature that $h_{n,j}(x_1, \dots, x_{j-1}) = x_{\iota_n(j)}$ (for $x_0 := 0 \leq x_1 \leq \dots \leq x_{j-1} \leq 1$ and $j > 1$), where $\iota_n(j) \in \{0, \dots, j-1\}$ is some threshold value of I_j, and $h_{n,1} = 0$. Another interesting possibility is to consider the class $\mathcal{M}_n$ of *memoryless rules* of the form

$$\tau = \min\{j :\ X_j \leq f_j\}, \tag{3}$$

where $(f_{n,j},\ j \in [n])$ is an increasing sequence of thresholds. These rules are again of the form (2), this time with constants in the role functions $h_{n,j}$. By familiar monotonicity arguments (which we recall in Section 4) the limiting value $V_\infty(\mathcal{M}) := \lim_{n\to\infty} V_n(\mathcal{M})$ (finite or infinite) exists for arbitrary q. See [18, 19] for other classes of stopping rules with restricted dependence on history.

Memoryless rules were intensively studied in the context of Robbins' problem, in which case they outperform, asymptotically, the rank rules, meaning that $V_\infty(\mathcal{M}) < V_\infty(\mathcal{R})$, see [1, 4, 5]. In a recent survey of Robbins' problem Bruss [3] stressed that a principal further step would be to either prove or disprove that $V_\infty(\mathcal{F}) < V_\infty(\mathcal{M})$. Coincidence of the asymptotic values $V_\infty(\mathcal{F}) = V_\infty(\mathcal{M})$ would imply that history dependence of the overall optimal rule were negligible, meaning that deciding about some X_j one should essentially focus on the current observation alone.

In this paper we extend the approach in [11, 12, 13, 14] by establishing that the stopping problem in $\mathcal{F}_n$ has a limiting '$n = \infty$' form based on the planar Poisson process. The interpretation of limit risks in terms of the infinite model makes obvious the inequality $V_\infty(\mathcal{F}) < V_\infty(\mathcal{M})$ for any q provided the values are finite, which is true for both the best-choice problem and Robbins' problem. Thus the complexity does

not disappear in the limit, and the full history dependence persists. The finiteness is guaranteed if $q(r)$ does not grow too fast, e.g. $q(r) < c\exp(r^\beta)$ $(0 < \beta < 1)$ is enough. In connection with Robbins' problem, the limiting form was reported by the author at the INFORMS Conference on Applied Probability (Atlanta, 14-16 June 1995), although the Poisson embedding had been exploited earlier [6] in the analysis of rank rules. See [15] for a similar development in the problem of minimising $\mathbb{E}[X_\tau]$.

2. A model based on the planar Poisson process Throughout we shall use the notation $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, and $\overline{\mathbb{R}}_+ = [0, \infty]$ for the compactified halfline.

Let $\mathcal{P}$ be the scatter of atoms of a homogeneous Poisson point process in the strip $[0,1] \times \overline{\mathbb{R}}_+$, with the intensity measure being the Lebesgue measure $\mathrm{d}t\mathrm{d}x$. The infinite collection of atoms can be labelled $(T_1, X_{1,1}), (T_2, X_{1,2}), \ldots$ by increase of the second component. Thus $\boldsymbol{X}_1 := (X_{1,1}, X_{1,2}, \ldots)$ is the increasing sequence of points of a unit Poisson process on $\mathbb{R}_+$, the T_r's are iid uniform $[0,1]$, and $\boldsymbol{X}_1$ and $(T_r,\ r = 1, 2, \ldots)$ are independent. An atom $(T_r, X_{1,r}) \in \mathcal{P}$ will be understood as observation with value $X_{1,r}$, arrival time T_r and *final rank* r. We define the *initial rank* of $(T_r, X_{1,r})$ as one plus the number of atoms in the open rectangle $]0, T_r[\, \times\,]0, X_{1,r}[$. Note that the coordinate-wise ties among the atoms only have probability zero.

To treat in a unified way both finite and infinite point configurations in the strip, we introduce the space $\mathcal{X}$ of all nondecreasing nonnegative sequences $\boldsymbol{x} = (x_1, x_2, \ldots)$ where $x_r \in \overline{\mathbb{R}}_+$, with the convention that a sequence with finitely many proper terms is always padded by infinitely many terms ∞. In particular, the sequence $\varnothing := (\infty, \infty, \ldots)$ is the sequence with no finite terms. The space $\mathcal{X}$ is endowed with the product topology inherited from $\overline{\mathbb{R}}_+^\infty$. We denote $\boldsymbol{x} \cup x$ the nondecreasing sequence obtained by inserting $x \in \overline{\mathbb{R}}_+$ in $\boldsymbol{x}$, with understanding that $\boldsymbol{x} \cup \infty = \boldsymbol{x}$. A strict partial order on $\mathcal{X}$ is defined by setting $\boldsymbol{x} \prec \boldsymbol{y}$ if $x_r \leq y_r$ for $r = 1, 2, \ldots$ with at least one of the inequalities strict. Clearly, $\boldsymbol{x} \cup x \prec \boldsymbol{x}$ for $x < \infty$.

We regard $\boldsymbol{X}_1$ as the terminal state of a $\mathcal{X}$-valued process $(\boldsymbol{X}_t, t \in [0,1])$, where $\boldsymbol{X}_t$ is obtained by removing the entries $X_{1,r}$ of $\boldsymbol{X}_1$ with $T_r > t$. Clearly, $\boldsymbol{X}_t$ is an increasing sequence of atoms of a Poisson process on $\mathbb{R}_+$ with intensity measure $t\mathrm{d}x$. For $t \in \{T_r\}$ let X_t, R_t, I_t be the value, the final rank and the initial rank of the observation arrived at time t, respectively, and for $t \notin \{T_r\}$ let $X_t = R_t = I_t = \infty$. We have $\boldsymbol{X}_t = \boldsymbol{X}_{t-} \cup X_t$, so $\boldsymbol{X}_t = \boldsymbol{X}_{t-}$ unless $t \in \{T_r\}$.

The process $(\boldsymbol{X}_t,\ t \in [0,1])$ is Markovian, with right-continuous paths, the initial state $\boldsymbol{X}_0 = \varnothing$ and the jump-times $\{T_r\}$ which comprise a dense subset of $[0,1]$. Each component $(X_{t,i},\ t \in [0,1])$ is a nonincreasing process, which satisfies $X_{0+,i} = \infty$ and changes its value at every *i-record* (observation of initial rank i). The jump-times of $(\boldsymbol{X}_{t,i},\ t \in [0,1])$ are the arrival times of i-records; these occur according to a Poisson process of intensity $t^{-1}\mathrm{d}t$ independently for distinct $i \in \mathbb{N}$, as is known from the extreme-value theory.

Define a *stopping rule* τ to be a variable which may only assume one of the random values $\{T_r\} \cup \{1\}$, and satisfies the measurability condition $\{\tau \leq t\} \in \sigma(\boldsymbol{X}_s,\ s \leq t)$ for $t \in [0,1]$. The condition says that the decision to stop not later than t is determined by atoms $\mathcal{P} \cap ([0,t] \times \mathbb{R}_+)$ arrived within the time interval $[0,t]$. Such rules are called in [15, Definition 2.1] 'canonical stopping times'.

We fix a nondecreasing nonnegative loss function q satisfying $q(1) < q(\infty)$. The

risk incurred by stopping rule τ is assumed to be

$$\mathbb{E}[q(R_\tau)] = \sum_{r=1}^{\infty} q(r)\ \mathbb{P}(\tau = T_r) + q(\infty)\ \mathbb{P}(\tau = 1), \tag{4}$$

where the terminal component is nonzero if and only if $\mathbb{P}(\tau = 1) > 0$. Let $\mathcal{F}$ be the set of all stopping rules, and let $V(\mathcal{F}) = \inf_{\tau \in \mathcal{F}} \mathbb{E}[q(R_\tau)]$ be the minimal risk.

The class $\mathcal{R}$ of *rank rules* is defined by a more restrictive measurability condition $\{\tau \leq t\} \in \sigma(I_s,\ s \leq t)$ for $t \in [0,1]$. That is to say, by a rank rule the information of observer at time t amounts to the collection of arrival times on $[0,t]$ of i-records, for all $i \in \mathbb{N}$. The optimal stopping problem in $\mathcal{R}$ is equivalent to 'the infinite secretary problem' in [10]. By [10, Theorem 4.1] there exists an optimal rank rule of the form $\tau = \inf\{t : I_t \leq \iota(t)\}$ $(\inf \varnothing = 1)$, where $\iota : [0,1[\to \mathbb{N} \cup \{0\}$ is a nondecreasing function. For instance, in the best-choice problem $\iota(t) = \mathbf{1}(t \geq e^{-1})$.

A *memoryless* rule is a stopping rule of the form

$$\tau = \inf\{t : X_t \leq f(t)\} \qquad (\text{with } \inf \varnothing = 1), \tag{5}$$

where $f : [0,1[\to \mathbb{R}$ is a nondecreasing function. Denote $\mathcal{M}$ the class of memoryless rules, and denote $V(\mathcal{M}) = \inf_{\tau \in \mathcal{M}} \mathbb{E}[q(R_\tau)]$ its stopping value. One could consider a larger class of stopping rules by which the decision to stop depends only on the current observation. However, the following lemma, analogous to [1, Lemma 2.1], shows that such extension of $\mathcal{M}$ does not reduce the risk.

Lemma 1. *Let $A \subset [0,1] \times \mathbb{R}_+$ be a Borel set. For the stopping rule $\tau = \inf\{t :\ (t, X_t) \in A\}$ there exists a memoryless rule whose expected loss is not larger than that of τ.*

Proof. It is sufficient to consider sets A such that the area of $A \cap ([0,t] \times \mathbb{R}_+)$ is finite for every $t < 1$. Indeed, if the area of $A \cap ([0,t] \times \mathbb{R}_+)$ is infinite for some $s < 1$ then $\tau < s$ a.s., hence letting A' to be $A \cap ([0,s] \times \mathbb{R}_+)$ shifted by $1 - s$ to the right we obtain a rule not worse than τ. Replace each vertical section of A by an interval adjacent to 0 of the same length, thus obtaining subgraph of a function g. This preserves the distribution of the stopping rule and does not increase the risk, by the monotonicity of q. Break $[0,1]$ into intervals of equal size δ and approximate g (in L^1) by a right-continuous function g_δ, constant on these intervals. Suppose on some adjacent intervals $[t, t+\delta[,\ [t+\delta, t+2\delta[$ we have $g_\delta(t) > g_\delta(t+\delta)$. Let g'_δ be another piecewise constant function with exchanged values on these intervals, $g_\delta(t+\delta)$ and $g_\delta(t)$, but outside $[t,\ t+2\delta]$ coinciding with g. Let $\mathcal{P}'$ be the scatter of atoms obtained by exchanging the strips $[t, t+\delta[\ \times \mathbb{R}_+$ and $[t+\delta, t+2\delta[\ \times \mathbb{R}_+$. Obviously, $\mathcal{P}' \stackrel{d}{=} \mathcal{P}$. To compare two stopping rules τ and τ' defined as in (5), but with g_δ, respectively g'_δ, in place of f, we consider the selected atom (τ, X_τ) as a function of $\mathcal{P}$, and consider $(\tau', X_{\tau'})$ as a function of $\mathcal{P}'$. It is easy to see that $X_\tau = X_{\tau'}$ unless $([t+\delta, t+2\delta[\ \times [0, g(t+\delta)]) \cap \mathcal{P} \neq \varnothing$, whereas in the latter case $X_{\tau'}$ is stochastically smaller than X_τ. The advantage comes from the event that each of the strips contains an atom below the graph of g_δ. It follows that τ' does better. Iterating this exchange argument, we see that the rule defined by g_δ is improved by a memoryless rule with a piecewise constant function. Letting $\delta \to\ 0$ shows that one can reduce A to a subgraph of a monotonic $f : [0,1[\to \overline{\mathbb{R}}_+$. $\square$

Given the initial rank $I_t = i$ and the value $X_t = x$ of some observation at time t, the final rank of the atom (t, x) is i plus the number of atoms south-east of (t, x), the latter being a Poisson variable with parameter $\bar{t}x$, where and henceforth

$$\bar{t} := 1 - t.$$

By independence properties of $\mathcal{P}$, the adapted loss incurred by stopping at (t, x) is equal to $Q(\bar{t}x, i)$, where

$$Q(\xi, i) := \sum_{r=i}^{\infty} q(r)\, e^{-\xi} \frac{\xi^{r-i}}{(r-i)!} \tag{6}$$

For instance, $Q(\bar{t}x, i) = 1 - e^{-\bar{t}x}\mathbf{1}(i = 1)$ in the best-choice problem, and $Q(\bar{t}x, i) = \bar{t}x + i$ in Robbins' problem. The formula for Q is extended for infinite values of the arguments as $Q(\cdot, \infty) = Q(\infty, \cdot) = q(\infty)$. It is seen from the identity

$$\frac{\mathrm{d}\left[e^{\xi}Q(\xi, 1)\right]}{\mathrm{d}\xi^{i-1}} = e^{\xi}Q(\xi, i)$$

that the series $Q(\cdot, i)$ have the same convergence radius for all i.

3. Memoryless rules and finiteness of the risk For τ a memoryless rule (5) with monotone f, denote $L(f) = \mathbb{E}\,[q(R_\tau)]$ the expected loss. Introduce the integrals

$$F(t) = \int_0^t f(s)\,\mathrm{d}s\,, \qquad S(x) = \int_0^x f^{-1}(y)\,\mathrm{d}y = xf^{-1}(x) - F(f^{-1}(x))\,,$$

where f^{-1} is the right-continuous inverse with $f^{-1}(x) = 0$ for $x < f(0)$. Note that $\mathbb{P}(\tau > t) = \exp(-F(t))$, and that given $\tau = t < 1$ the law of X_τ is uniform on $[0, f(t)]$. The formula for the risk follows by conditioning on the location of the leftmost atom below the graph of f and using the fact that the configurations of atoms above the graph and below it are independent:

$$L(f) = \int_0^1 e^{-F(t)}\,\mathrm{d}t \int_0^{f(t)} Q(\bar{t}x + S(x), 1)\,\mathrm{d}x\; + e^{-F(1)}\,q(\infty). \tag{7}$$

Assuming that $F(1) = \infty$, so the terminal part is 0, computation of the first variation of $L(f)$ shows that an optimal f must satisfy a rather complicated functional equation:

$$Q(f(t) - F(t), 1) = \tag{8}$$
$$\int_t^1 \exp(F(t) - F(s))\mathrm{d}s \left[\int_0^{f(t)} Q(S(x) + x\bar{s}, 1)\mathrm{d}x + \int_{f(t)}^{f(s)} Q(S(x) + x\bar{s}, 2)\mathrm{d}x\right].$$

A rough upper bound

$$L(f) \le \int_0^1 e^{-F(t)}\,\mathrm{d}t \int_0^{f(t)} Q(x, 1)\,\mathrm{d}x\; + e^{-F(1)}\,q(\infty) \tag{9}$$

follows from $\bar{t}x + S(x) \le x$.

The bound (9) is computable for the loss functions

$$q(r) = (r-1)(r-2)\cdots(r-\ell) \qquad (\ell \in \mathbb{N}), \tag{10}$$

in which case we have a very simple formula $Q(\xi, 1) = \xi^\ell$, and (9) becomes

$$L(f) \le (\ell+1)^{-1} \int_0^1 e^{-F(t)} f(t)^{\ell+1} \mathrm{d}t\,.$$

Solving the variational problem for F with boundary conditions $F(0) = 0,\ F(1) = \infty$, we see that the minimal value of the right-hand side is $(\ell+1)^\ell$, which is attained by the function $f(t) = (\ell+1)/(1-t)$.

It is instructive to directly analyse the memoryless rules with hyperbolic threshold

$$f_b(t) := \frac{b}{1-t} \qquad (b > 0)$$

and q as in (10). We calculate $e^{-F(t)} = (1-t)^b$ and $S(x) = (x - b - b\log(x/b))$ (for $x > f(0) = b$). For $\ell = 1$ integrating by parts in (7) we obtain

$$L(f_b) = \frac{b}{2} + \frac{1}{b^2-1}, \tag{11}$$

which is finite for all $b > 1$, with the minimum $1.3318\cdots$ attained at $b = 1.9469\cdots$ (which agrees with [1, Example 4.2] where the minimum is $2.3318\cdots$ for the linear loss $q(r) = r$). For $\ell = 2$

$$L(f_b) = \frac{b^3}{3} + \frac{2(b^4 - 2b^3 + 2b^2 + 6b - 4)}{(b-2)(b-1)^2(b+1)(b+2)}, \tag{12}$$

which is finite for all $b > 2$, with minimum $4.4716\cdots$ at $b = 2.96439\cdots$. Formulas become more involved for larger ℓ, a common feature being that $L(f_b) < \infty$ for $b > \ell$. For $\ell = 3$, the minimum is 24.8061 at $3.9734\cdots$. For $\ell = 4$, the minimum is $194.756\cdots$ at $b = 4.979\cdots$. The upper bound (9) becomes

$$L(f_b) < \int_0^1 (1-t)^b \int_0^{b/(1-t)} x^\ell \mathrm{d}x = \frac{b^{\ell+1}}{(\ell+1)(b-\ell)},$$

which attains minimum at $b = \ell+1$ in agreement with what we have obtained above.

Remark. Notably, the memoryless rule with threshold $f_{\ell+1}$ is overall optimal in the related stopping problem $\mathbb{E}[(X_\tau)^\ell] \to \inf$, for arbitrary $\ell > 0$. For $\ell = 1$ we face here a variant of 'Moser's problem' associated with $\mathcal{P}$ (see [1, 3, 15] and references therein).

It can be read from [3, 1, 7] that for the linear loss $q(r) = r$ we have $V(\mathcal{M}) = \inf L(f) < V(\mathcal{R}) = 3.8695\cdots$.

The minimiser of $L(f)$ is not known explicitly, but some approximations to it can be read from [1] (where they appear in the course of asymptotic analysis of the finite-n Robbins' problem). We did not succeed to solve (8) even for the best choice problem,

although there is a simple suboptimal rule with constant threshold $f(t) = 1.503\cdots$ achieving $L(f) = 1 - 0.517\cdots$ (to be compared with the value $V(\mathcal{F}) = 1 - 0.580\cdots$, see [11, p. 682]) hence beating the rank rules: $V(\mathcal{M}) < V(\mathcal{R}) = 1 - 0.368\cdots$.

It would be interesting to know for which q the memoryless rules outperform the rank rules and if it is possible, for unbounded q, to have the memoryless risk finite while infinite for the rank rules. We sketch some results in this direction. From the above elementary estimates $V(\mathcal{M}) < \infty$ provided $q(r) < c\, r^\ell$ for some constants $c > 0$, $\ell > 0$. For such q the risk of rank rules is also finite. Moreover, Mucci [17, p. 426] showed that for the loss function $q(r) = r(r+1)\cdots(r+\ell-1)$ $(\ell \in \mathbb{N})$ the minimum risk of rank rules is

$$V(\mathcal{R}) = \ell! \prod_{j=1}^{\infty} \left(1 + \frac{\ell+1}{j}\right)^{\ell/(\ell+j)}$$

(which extends the $\ell = 1$ result from [7]). For $\ell = 2$ the formula yields $33.260\cdots$, while the f_b-rules do worse, with $\inf_b L(f_b) = 38.068\cdots$ (as computed from (11) and (12) using the linearity of $L(f)$ in q).

In fact, $V(\mathcal{M}) < \infty$ for many loss fuctions growing much faster than polynomials.

Proposition 2. *If $q(r) < c\exp(x^\beta)$ for some $c > 0$ and $0 < \beta < 1$ then $V(\mathcal{M}) < \infty$.*

Proof. The risk is finite for the memoryless rule with $f(t) = (1-t)^{-\alpha}$ for any $\alpha > (1-\beta)^{-1}$. To see this, use the bound (9) and formulas

$$Q(x,1) = O(\exp(x^\beta)) \quad (x \to \infty), \qquad \exp(-F(t)) = \exp\left(-\frac{1}{(\alpha-1)(1-t)^{\alpha-1}}\right),$$

which also imply that for this rule $\mathbb{P}(\tau = 1) = 0$. Now $\mathbb{E}[\exp((X_\tau)^\beta))]$ is estimated from asymptotics of the incomplete gamma function. □

However, the risk is infinite for any stopping rule if q grows too fast. The following result is an analogue of [10, Proposition 5.3] for rank rules.

Proposition 3. *If $Q(b,1) = \infty$ for some $b \in \mathbb{R}_+$ then $V(\mathcal{F}) = \infty$, i.e. there is no stopping rule $\tau \in \mathcal{F}$ with finite risk.*

Proof. Choose any x with $S(x) = x - b - b\log(x/b) > b$. The conditional loss by stopping above f_b is infinite, thus we can only consider stopping rules τ which never do that and satisfy $\mathbb{P}(\tau = 1) = 0$. On the other hand, on the nonzero event $\{\mathcal{P} \cap \{(t,y) : y < \min(x, f(t))\} = \varnothing\}$ stopping occurs at some atom (s,z) with $s > 1 - b/x$, $z > x$, and averaging we see that the expected loss is infinite. □

Remark By [10, Section 5], $V(\mathcal{R}) = \infty$ if $\sum_r (\log q(r))/r^2 = \infty$. For instance, the loss structure $q(r) = e^r$ implies that the risk of rank rules is infinite. It is not known if the risk of rank rules is finite for $q(r) = \exp(x^\beta)$ with $0 < \beta < 1$.

For the sequel we assume that the loss function satisfies

$$\limsup \frac{q(r+1)}{q(r)} = C, \tag{13}$$

with some constant $C > 0$. The assumption implies that $Q(x, i) < \infty$ for all finite x, i. Another consequence is that $\mathbb{E}[q(R_\tau)] < \infty$ implies $\mathbb{E}[q(R_\tau + N)] < \infty$ for N either a fixed positive integer or a Poisson random variable, independent of τ.

Lemma 4. *If $\mathbb{E}[q(R_\tau) \,|\, \boldsymbol{X}_0 = \boldsymbol{x}] < \infty$ then $\mathbb{E}[q(R_\tau) \,|\, \boldsymbol{X}_0 = \boldsymbol{x}']$ is finite and continuous in x, where $\boldsymbol{x}'$ is either $\boldsymbol{x} \cup x$ or $(x_1 + x, x_2 + x, \ldots)$.*

Proof. As x changes to some x', the outcome R_τ can only change if there is an atom between x and x', which occurs with probability about $|x - x'|$ when x, x' are close. Conditionally on this event, the change of expected loss is bounded in consequence of (13). □

3. Properties of the optimal rule The optimal stopping problem in $\mathcal{F}$ is a problem of Markovian type, associated with the time-homogeneous Markov process $((\boldsymbol{X}_t, I_t),\ t \in [0, 1])$, with state-space $\mathcal{X} \times \overline{\mathbb{N}}$ and time-dependent loss $Q(\bar{t}X_t, I_t)$ for stopping at time t. If I_t assumes some finite value i then $t \in \{T_r\}$ and $X_{t,i} = X_t$, which combined with the fact that ranking of the arrivals after t depends on $\mathcal{P} \cap ([0, t] \times \mathbb{R}_+)$ through $\boldsymbol{X}_t$ shows that $(\boldsymbol{X}_t, I_t)$ indeed summarises all relevant information up to time t. We choose $(\boldsymbol{X}_t, I_t)$ in favour of (probabilistically equivalent) data $(\boldsymbol{X}_{t-}, X_t)$ since x_i is well-defined as a function of $(\boldsymbol{x}, i)$ even if $\boldsymbol{x}$ has repetitions.

Following a well-known recipe, we consider a family of conditional stopping problems parametrised by $(t, \boldsymbol{x})$. This corresponds to the class of stopping rules $\tau > t,\ \tau \in \mathcal{F}$ that operate under the condition $\boldsymbol{X}_t = \boldsymbol{x}$. The effect of the conditioning is that each $x_r < X_\tau$ contributes one unit to R_τ in the event $\tau < 1$. The variable t can be eliminated by a change of variables which exploits the *self-similarity* of $\mathcal{P}$ (a property which has no analogue in the finite-n setting): for $t \in]0, 1[$ fixed, the affine mapping $(s, x) \mapsto ((s - t)/\bar{t}, x\bar{t})$ preserves both the coordinate-wise order and the Lebesgue measure, hence transforms the point process $\mathcal{P} \cap ([t, 1] \times \mathbb{R}_+)$ into a distributional copy of $\mathcal{P}$ with the same ordering of the atoms. Thus we come to the following conclusion:

Lemma 5. *The stopping problem from time t on with history $\boldsymbol{x}$ is equivalent to the stopping problem starting with $\boldsymbol{X}_0 = \bar{t}\boldsymbol{x}$ at time 0.*

Let $v(\boldsymbol{x})$ be the minimum risk given $\boldsymbol{X}_0 = \boldsymbol{x}$. The function v, defined on the whole of $\mathcal{X}$, satisfies a lower bound

$$v(\boldsymbol{x}) \geq \sum_{r=1}^{\infty} q(r)(e^{-x_{r-1}} - e^{-x_r}) \qquad (x_0 = 0), \tag{14}$$

which is strict if the series converges (the bound is a continuous-time analogue of the finite-n 'half-prophet' bounds in [4, Lemma 3.2]). The bound follows by observing that X_τ cannot exceed the smallest value arrived on $[0, 1]$.

If $V(\mathcal{F}) = \infty$ then, of course, $v(\boldsymbol{x}) = \infty$ everywhere, but for arbitrary unbounded q there exists a dense in $\mathcal{X}$ set of sequences $\boldsymbol{x} = (x_r)$ for which $x_r \uparrow \infty$ so slowly that $v(\boldsymbol{x}) = \infty$. Thus if $q(\infty) = \infty$, the function v is discontinuous at every point where it is finite. If q is truncated at m, then clearly v depends only on the first $m - 1$ components of $\boldsymbol{x}$ and satisfies $v(\boldsymbol{x}) < q(m)$. Let $\mathbf{0} = (0, 0, \ldots)$.

Lemma 6. *The following hold:*

(i) $v(\boldsymbol{x}) < \infty$ *implies that* $v(\boldsymbol{x} \cup x)$ *is finite and continuous in* x*,*

(ii) *if* $q(\infty) < \infty$ *then* v *is continuous, and satisfies* $v(\boldsymbol{x}) < q(\infty)$ *for* $x_1 > 0$*.*

(iii) $v(\boldsymbol{x}) \to q(\infty)$ *as* $\boldsymbol{x} \to \mathbf{0}$*.*

Proof. Let τ be ϵ-optimal under the initial configuration $\boldsymbol{x} \cup x$. Applying τ under $\boldsymbol{x} \cup x'$, Lemma 4 implies that $v(\boldsymbol{x} \cup x') \leq v(\boldsymbol{x} \cup x) + \epsilon$. Changing the roles of x, x' and letting $\epsilon \to 0$ yield (i). The continuity of v follows directly from (i) if q is truncated at some m. The general bounded case follows by approximation as $m \to \infty$. Assertion (iii) can be derived from (14). □

Lemma 7. *If* q *is not truncated then*

(i) $Q(x, i)$ *is strictly increasing in both* x *and* i*,*

(ii) $\boldsymbol{x} \prec \boldsymbol{y}$ *implies* $v(\boldsymbol{x}) < v(\boldsymbol{y})$ *provided these are finite,*

If q *is truncated at* m *and* $q(m-1) < q(m)$ *then* (i) *is valid only for* $i \in [m]$*,* $Q(\boldsymbol{x}, i) = q(m) = q(\infty)$ *for* $i \geq m$*, and a counterpart of* (ii) *holds for the order defined on the first* $m-1$ *components, with* $v(\boldsymbol{x}) < q(m)$ *for all* $\boldsymbol{x} \in \mathcal{X}$ *with* $x_{m-1} > 0$*.*

Proof. Assertion (i) follows from (6) and the monotonicity of q. For (ii), observe that $\boldsymbol{x} \prec \boldsymbol{y}$ implies $\#\{i : x_i < x\} \geq \#\{i : y_i < x\}$ for all $x > 0$. Hence for every rule τ the stopped final rank under $\boldsymbol{X}_0 = \boldsymbol{x}$ cannot increase when the condition is replaced by $\boldsymbol{X}_0 = \boldsymbol{y}$.

□

Let $i(\boldsymbol{x}, x) := \#\{r : x_r \leq x\}$ and suppose $\boldsymbol{x}$ satisfies $0 < x_1 \leq x_2 \leq \ldots \leq \infty$. Applying Lemma 7, we see that if q is not truncated then the function $Q(x, i(\boldsymbol{x}, x))$ is strictly increasing in x from $q(1)$ to $q(\infty)$. If q is truncated at m and $q(m-1) < q(m)$ then $Q(x, i(\boldsymbol{x}, x))$ is strictly increasing as x varies from 0 to x_{m-1}, with $Q(x, i(\boldsymbol{x}, x)) = q(m)$ for $x \geq x_{m-1}$. On the other hand, $(\boldsymbol{x} \cup x) \prec (\boldsymbol{x} \cup y)$ for $x < y$, hence $v(\boldsymbol{x} \cup x)$ is nonincreasing in x. Thus introducing

$$h(\boldsymbol{x}) := \sup\{x : Q(x, i(\boldsymbol{x}, x)) < v(\boldsymbol{x} \cup x)\}$$

we have $Q(x, i(\boldsymbol{x}, x)) < v(\boldsymbol{x} \cup x)$ for $x < h(\boldsymbol{x})$, and $Q(x, i(\boldsymbol{x}, x)) \geq v(\boldsymbol{x} \cup x)$ for $x \geq h(\boldsymbol{x})$. Subject to obvious adjustments, the definition of $h(\boldsymbol{x})$ makes sense for every $\boldsymbol{x} \neq \mathbf{0}$ in the untruncated case, and for $x_{m-1} > 0$ in the truncated.

We are ready to show that memoryless rules are not optimal.

Proposition 8. *If* $V(\mathcal{F}) < \infty$ *then* $V(\mathcal{F}) < V(\mathcal{M})$*.*

Proof. For a memoryless rule with threshold function f to be optimal, we must have $v(\bar{t}\boldsymbol{X}_t) < Q(\bar{t}X_t, i(\boldsymbol{X}_{t-}, X_t))$ for $X_t > f(t)$, and $v(\bar{t}\boldsymbol{X}_t) > Q(\bar{t}X_t, i(\boldsymbol{X}_{t-}, X_t))$ for $X_t < f(t)$, because otherwise the rule can be improved. This forces $f(t) = h(\bar{t}\boldsymbol{x})$, which does not hold since h is not constant.

To demonstrate concretely how a memoryless rule with threshold f can be improved let us apply the same idea as in [4, Section 5]. Assume $q(\infty) = \infty$. Suppose (t, x) is above the graph of f, hence should be skipped by the memoryless

rule. Let $i = i(\boldsymbol{x}, x)$ be the initial rank under history $\boldsymbol{x}$. Varying finitely many of the components x_r $(r > i)$ we can achieve that the bound (14) be arbitrarily large while the expected loss of stopping remains unaltered $Q(\bar{t}x, i)$. For such $\boldsymbol{x}$ we have $v(\bar{t}(\boldsymbol{x} \cup x)) > Q(\bar{x}, i(\boldsymbol{x}, x))$ hence stopping strictly reduces the risk on some event of positive probability. □

Based on the function $h : \mathcal{X} \to \overline{\mathbb{R}}_+$, we construct a predictable process

$$H_t := h(\boldsymbol{X}_{t-} \setminus \{X_{1,r} : T_r < t,\ X_{1,r} < h(\boldsymbol{X}_{T_r -})\}) \qquad (t \in [0,1]).$$

Let $\boldsymbol{Y}_t$ be a thinned sequence obtained by removing the terms in $\{\cdots\}$ from $\boldsymbol{X}_{t-}$, so $H_t = h(\boldsymbol{Y}_t)$. Intuitively, H_t is a history-dependent threshold which depends on the configuration of atoms $\boldsymbol{X}_{t-}$ that arrived on $[0, t[$ and are above the curve $(H_s,\ s \in [0, t[)$. As t starts increasing from 0, the process H_t coincides with $h(\boldsymbol{X}_{t-})$ as long as there are no atoms below the threshold, while at the first moment this occurs the atom is discarded, and does not affect the future path of the process.

Remark The reason for thinning $\mathcal{P}$ is that we wish to see (H_t) as an increasing process defined for all t, as opposed to considering $h(\boldsymbol{X}_{t-})$ killed as soon as the threshold is undershoot.

We list some properties of (H_t) which follow directly from the definition and Lemmas 6 and 7 (under $\boldsymbol{X}_0 = \varnothing$).

Lemma 9. (i) (H_t) *is nondecreasing on* $[0, 1[$ *.*

(ii) *If* $V(\mathcal{F}) < \infty$ *then* H_0 *is the unique root of* $Q(x, 1) = v(x \cup \boldsymbol{\infty})$*.*

(iii) $H_{1-} = Y_{1,m-1}$ *if* q *is truncated at* m *and* $q(m-1) < q(m)$*.*

(iv) $H_{1-} = \infty$ *if* q *is not truncated.*

To gain some intuition about the behaviour of (H_t) we shall gradually increase the complexity of loss function. In the simplest instance of the best-choice problem, v depends only on x_1 (see [12, Equations (8) and (13)]) and there is an explicit formula for threshold

$$H_t = \min(f_b(t), Y_{t,1}) \qquad (b = 0.804\cdots).$$

That is to say, as t starts increasing from 0, H_t is a deterministic *drift* process until it hits the level of the lowest atom above the graph. The drift is hyperbolic due to self-similarity of $\mathcal{P}$ (Lemma 5). After this random time, H_t has a *flat*, which appears because it is never optimal to stop at observation with initial rank 2 or larger. On the first part of the path H_t satisfies $Q(H_t, 1) = v(\bar{t}(\boldsymbol{Y}_t \cup H_t))$, and on the second $Q(H_t, 1) < v(\bar{t}(\boldsymbol{Y}_t \cup H_t))$.

If q is strictly truncated at $m = 3$, meaning that $q(2) < q(3) = q(\infty)$, a new effect appears. For t sufficiently small, as long as $H_t < Y_{t,1}$ each 1-record above the threshold causes a *jump*, because $v(\bar{t}\boldsymbol{Y}_t)$ jumps and the threshold must go up to compensate. Thus (H_t) has both drift and jump components. The jump locations are the 1-record times accumulating near 0 at rate $t^{-1}\mathrm{d}t$. As H_t hits $Y_{t,1}$, there is a

possible flat, then a period of deterministic drift where $Q(H_t, 2) = v(\bar{t}(\boldsymbol{Y}_t \cup H_t))$, and finally there is a flat at some level $Y_{t,2}$ (then $Y_{t,2} = Y_{1,2}$).

For q strictly truncated at $m > 3$, the jump locations are included in $m-2$ record-time processes of atoms with initial rank at most $m-2$, there are $m-1$ potential flats and a drift component between the flats. We do not assert that the number of flats is always exactly $m-1$, because it is not at all clear if (H_t) can break a level $Y_{t,r}$ for $r < m-1$ by jumping through it, hence sparing a flat.

Now suppose that q is not truncated and that $H_t < \infty$ everywhere on $[0, 1[$ with probability one. Then, outside the union of flat intervals, *every* arrival above H_t causes a jump, thus the set of jump locations is dense there. The number of flats may be infinite, and outside the flats $Q(H_t, i(\boldsymbol{Y}_t, H_t)) = v(\bar{t}(\boldsymbol{Y}_t \cup H_t))$.

In the case of Robbins' problem, we have by linearity of the loss $Q(x, i+1) - Q(x, i) = 1$ and $v(\boldsymbol{x} \cup x) - v(\boldsymbol{x}) < 1$ (if $v(\boldsymbol{x} \cup x) < \infty$). Thus $Q(x, i(\boldsymbol{x}, x)) = v(\boldsymbol{x} \cup x)$ implies $Q(x, i(\boldsymbol{x}, x) + 1) > v(\boldsymbol{x} \cup x \cup x')$ for arbitrary x'. But this means that (H_t) cannot cross any $Y_{t,i}$ by a jump. It follows that (H_t) has infinitely many flats at all levels $Y_{1,r}$ ($r \in \mathbb{N}$). The presence of all three effects (drift, jumps and flats) and the lack of independence of increments property all leave a little hope for a kind of more explicit description of (H_t).

The optimality principle requires stopping at atom (t, x) when the history $\boldsymbol{X}_{t-} = \boldsymbol{x}$ satisfies $Q(\bar{t}x, i(\boldsymbol{x}, x)) < v(\bar{t}\boldsymbol{x})$, whence the following analogue of (2).

Proposition 10. *If $V(\mathcal{F}) < \infty$ then $H_t < \infty$ a.s. for all $t < 1$ and the stopping rule*

$$\tau^* := \inf\{t : X_t < H_t\} \qquad (\inf \varnothing = 1)$$

is optimal in $\mathcal{F}$.

Proof. For bounded q a general result [21, Theorem 3, p. 127] is applicable since the function $Q(x, i(\boldsymbol{x}, x))$ is bounded and continuous on $\mathcal{X} \times \mathbb{N}$.

Alternatively, for q truncated at some m one can use results of the optimal stopping theory for discrete-time processes. To fit exactly in this framework, focus on the sequences of i-records (for $i \leq m-1$) that arrive on $[\epsilon, 1]$, and then let $\epsilon \to 0$. The general bounded case follows in the limit $m \to \infty$.

For unbounded q we use another kind of truncation (analogous to that in [3, Section 4]). For m fixed, let $Q^{(m)}(x, i) = Q(x, \max(i, m))$ and consider the stopping problem with loss $Q^{(m)}(\bar{t}x, i(\boldsymbol{x}, x)$ for stopping at (t, x) with history $\boldsymbol{x}$. This corresponds to ranking x relative to at most m atoms before t, but fully accounting all future observations below x. In this problem it is never optimal to stop at atom with relative rank m or higher. Indeed, stopping at (t, x) with such rank can be improved by continuing and then exploiting any hyperbolic memoryless rule with $b < \bar{t}x$ (stopping is guaranteed before 1 since the subgraph of f_b has infinite area). By discrete-time methods, optimality of the rule $\tau^{(m)} = \inf\{t : X_t < H_t^{(m)}\}$ in the truncated problem is readily acquired, with a nondecreasing predictable process $(H_t^{(m)})$ defined through $h^{(m)}(\boldsymbol{x}) := \sup\{x : Q^{(m)}(x, i(\boldsymbol{x}, x)) < v^{(m)}(\boldsymbol{x} \cup x)\}$, where $v^{(m)}$ is the minimum loss analogous to v. Obviously, $Q^{(m)}(x, i(\boldsymbol{x}, x)), v^{(m)}(\boldsymbol{x})$ is nondecreasing in m.

A decisive property of this kind of truncation is that $Q^{(m)}(x, i) = Q(x, i)$ for $m \geq i$. This implies that $H_t^{(m)}$ is eventually nondecreasing in m and there exists a pointwise limit $H'_t = \lim_{m\to\infty} H_t^{(m)}$, which defines a legitimate stopping rule τ' as the time

of the first arrival under H'. Denote for shorthand $L(\tau) = \mathbb{E}[Q(X_\tau, I_\tau)], L^{(m)}(\tau) = \mathbb{E}[Q^{(m)}(X_\tau, I_\tau)]$ and denote $u, u^{(m)}$ the minimum risks (so $u = V(\mathcal{F})$). Trivially, $\lim_{m\to\infty} u^{(m)} \le u$. On the other hand, by monotone convergence $L^{(m)}(\tau') \uparrow L(\tau) \ge u$. If follows that $u^{(m)} \le u$ and τ' is optimal. The convergence $v^{(m)}(\boldsymbol{x}) \uparrow v(\boldsymbol{x})$ is shown in the same way, from which $H'_t = H_t$ and $\tau' = \tau^*$ is optimal. □

Remark. Assumption (13) limits, by the virtue of Lemma 4, the risks of *all* stopping rules under various initial data, while we are really interested only in the properties of optimal or ϵ-optimal rules. We feel that Proposition 10 is still valid under the sole condition $V(\mathcal{F}) < \infty$, but history dependence makes proving this more difficult than in the analogous situation with rank rules [10].

As a by-product, we have shown that the risk in the truncated problem with loss function $q(\min(r, m))$ converges to $V(\mathcal{F})$. Indeed, the loss is squeezed between the loss in the modified truncated problem and the original untruncated loss.

From the formula for the distribution of the optimal rule,

$$\mathbb{P}(\tau^* > t) = \mathbb{E}\left[\exp\left(-\int_0^s H_s \mathrm{d}s\right)\right],$$

and arguing as in Lemma 1 we see that H_t cannot explode at some $t < 1$ if $V(\mathcal{F}) < \infty$.

The risk can be bounded from below in the spirit of (7) as

$$\mathbb{E}[q(R_{\tau^*})] \ge \mathbb{E}\left[\int_0^1 \exp\left(-\int_0^s H_s \mathrm{d}s\right)\int_0^{H_t} Q(\bar{t}x, \phi_H(x))\mathrm{d}x\right],$$

where $\phi_H(x)$ is the number of flats of (H_t) below x. If the loss function q has the property that the flats of (H_t) occur at all levels $X_{1,r},\ r \in \mathbb{N}$ (like in Robbins' problem) the equality holds. The same kind of estimate is valid for every stopping rule τ defined by means of an arbitrary nondecreasing predictable process like (H_t).

4. The infinite Poisson model as a limit of finite-n problems To connect the finite-n problem with its Poisson counterpart it is convenient to realise iid sequence in the following way [9, 11, 14]. Divide the strip $[0,1]\times\mathbb{R}_+$ in n vertical strips of the same width $1/n$. Let X_j be the atom of $\mathcal{P}$ with the lowest x-value. By properties of the Poisson process, $X_1, \ldots, X_n$ are iid with exponential distribution of rate $1/n$. Note that optimal stopping of $X_1, \ldots, X_n$ is equivalent to optimal stopping of $\mathcal{P}$ with the lookback option allowing the observer to return to any atom within a given $1/n$-strip (equivalently, at time $(j-1)/n$ to foresee the configuration of atoms up to time j/n). This embedding in $\mathcal{P}$ immediately implies $V_n(\mathcal{F}_n) < V(\mathcal{F})$. Moreover, as $n \to \infty$, each i-record process derived from $X_1, \ldots, X_n$ converges almost surely to the i-record process derived from $\mathcal{P}$. From this one easily concludes, first for truncated then for any bounded q, that $V_\infty(\mathcal{F}) = V(\mathcal{F})$, where $V_\infty(\mathcal{F}) = \lim_{n\to\infty} V_n(\mathcal{F}_n)$ as defined in Introduction.

For the general q, the relations

$$V_\infty(\mathcal{F}) = V(\mathcal{F}),\ V_\infty(\mathcal{R}) = V(\mathcal{R}),\ V_\infty(\mathcal{M}) = V(\mathcal{M})$$

follow (as in [1, 2, 4, 7, 9, 16]) from that in the truncated case, by combining monotonicity of risks in the truncation parameter m with the monotonicity in n stated in the next lemma.

Lemma 11. *$V_n(\mathcal{F}_n), V_n(\mathcal{R}_n), V_n(\mathcal{M}_n)$ are increasing with n.*

Proof. This all is standard, see the references above. We only add small details to [1, Theorem 2.4] for the $\mathcal{M}$-case. Let τ be an optimal memoryless rule in the problem of size $n+1$, and let τ' be a modified memoryless strategy which always skips the worst value $X_{n+1,n+1}$ but otherwise has the same thresholds as τ. (To apply τ' the observer must be able to recognise $X_{n+1,n+1}$ as it arrives.) Then τ' strictly improves τ in the event that τ stops at $X_{n+1,n+1}$. On the other hand, strategy τ' performs as a mixture of memoryless rules in the problem of size n, because given $X_{n+1,n+1} = x$ the other X_j's are iid uniform on $[0, x]$. Therefore $V_n(\mathcal{M}_n) < V_{n+1}(\mathcal{M}_{n+1})$. $\square$

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